ISRAEL JOURNAL OF MATHEMATICS 141 (2004), 119–123

A 2-GENERATED JUST-INFINITE PROFINITE GROUP WHICH IS NOT POSITIVELY GENERATED*

BY

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ABSTRACT

We construct a 2-generated profinite group which is just-infinite and not positively generated.

Since a profinite group G has a natural compact topology, it has also a Haar measure, which is determined uniquely by the algebraic structure of G. We normalize this measure so that G has measure 1, and it becomes a probability space. This allows us to define, for any positive integer k, $\mathrm{Prob}_G(k)$ as the measure of the subset $\{(g_1, \ldots, g_k) \in G^k \mid g_1, \ldots, g_k$ topologically generate $G\}$. A profinite group G is **positively finitely generated** (PFG) if for some k, the probability $\mathrm{Prob}_G(k)$ that k random elements generate G is positive.

Mann [6] proved that finitely generated prosoluble groups are PFG, as are the profinite completions of $\mathrm{SL}(d, \mathbb{Z})$ for $d \geq 3$. A stronger result has been proved in [2], where it is shown that if G is a finitely generated profinite group and there is a finite group which is not obtained as a quotient of an open subgroup of G, then G is PFG. Other examples of PFG groups occur in Bhattacharjee's work [1].

However, a free non-abelian profinite group is not PFG. Indeed, Kantor and Lubotzky proved that the direct product $\prod_n \mathrm{Alt}(n)^{n!/8}$, equipped with the product topology, is a finitely generated profinite group which is not PFG [5]. This

* This paper was mainly conceived during the workshop "Groups and Probability" organized in Budapest (June 30 – July 4, 2003) by the Erdős Center. Stimulating conversations during the conference led to this result and the author is grateful to both the organizers and the participants.
Received August 24, 2003

infinite direct product has as an epimorphic image the group $\prod_n \mathrm{Alt}(n)$, which is infinite and PFG (see [5], Proposition 14). In a recent meeting ("Groups and Probability", Budapest, June 30 – July 4, 2003) Pyber asked whether any profinite group which is not PFG behaves in this way and proposed the following conjecture: a finitely generated profinite group which is not PFG admits an infinite epimorphic image which is PFG. In this paper we give a counterexample to this conjecture; indeed, we construct a profinite group G which is 2-generated, non-PFG and just-infinite (i.e., any proper epimorphic image of G is finite).

Before starting with the construction of our example, we find it useful to recall a definition.

Definition 1 (see [4]): Let L be a finite monolithic primitive group and let M be its unique minimal normal subgroup. For each positive integer k, let L^k be the k-fold product of L. The **crown-based power** of L of size k is the subgroup L_k of L^k defined by

$$L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \bmod M\}.$$

Clearly, $\mathrm{soc}(L_k) = M^k$, $L_k/\mathrm{soc}(L_k) \cong L/M$ and the quotient group of L_k over any minimal normal subgroup is isomorphic to L_{k-1}, for $k > 1$. Moreover, it can be easily proved that L_k has the following property:

LEMMA 2: *Any normal subgroup of L_k either contains or is contained in $\mathrm{soc}(L_k)$.*

We shall construct a sequence $\{X_i\}_{i\in\mathbb{N}}$ of finite groups, satisfying the following properties:

(1) X_i is a crown-based power of a monolithic primitive group with nonabelian socle;
(2) $X_{i+1}/\mathrm{soc}(X_{i+1}) \cong X_i$;
(3) X_i can be generated by 2 elements.

In the sequel we will denote by $d(X)$ the smallest cardinality of a generating set of the group X. Before describing our construction we need to claim the following result:

PROPOSITION 3: *There exists an absolute constant γ with the following property. Suppose that L is a finite monolithic primitive group and $M = \mathrm{soc}\, L \cong S^n$ with S a nonabelian simple group. If $d(L/M) \leq 2$, then $d(L_t) = 2$ for any $t \leq \gamma|S|^n/n|\mathrm{Out}\, S|$.*

Proof: This is just a particular case of [3], Proposition 10. ∎

Now we can construct recursively our groups X_i. We start by setting $X_1 = \mathrm{Alt}(5)$. Assume that X_j has been defined for $1 \le j \le i$ and let $m_i = |X_i|$. We think of X_i as a regular permutation group of degree m_i and we take the wreath product $G_i = \mathrm{Alt}(m_i) \wr X_i$; note that G_i is a monolithic primitive group and $\mathrm{soc}(G_i) \cong (\mathrm{Alt}(m_i))^{m_i}$. Let

$$t_i = \left[\frac{\gamma(m_i!)^{m_i}}{m_i 2^{m_i+1}}\right];$$

we define $X_{i+1} = (G_i)_{t_i}$, i.e., X_{i+1} is the crown-based power of G_i of size t_i. By construction $\mathrm{soc}(X_{i+1}) \cong ((\mathrm{Alt}(m_i))^{m_i})^{t_i}$ and $X_{i+1}/\mathrm{soc}(X_{i+1}) \cong G_i/\mathrm{soc}(G_i) \cong X_i$. Moreover, by Proposition 3, $d(X_{i+1}) = 2$.

We have an epimorphism $\phi_{i+1,i}\colon X_{i+1} \to X_i \cong X_{i+1}/\mathrm{soc}(X_{i+1})$, and, for any $j > i$, the composition map $\phi_{ji} = \phi_{j,j-1} \cdots \phi_{i+1,i}$ is an epimorphism $X_j \to X_i$. So $\{X_i, \phi_{ji}\}$ is a projective system and we may consider the inverse limit $G = \varprojlim_{i\in\mathbb{N}} X_i$. For any $i \in \mathbb{N}$, there is a continuous epimorphism $\pi_i\colon G \to X_i$; let $N_i = \ker \pi_i$. We have that $N_{i+1} < N_i$ and $\{N_i\}_{i\in\mathbb{N}}$ is a fundamental system of open neighborhoods of the identity in G. More precisely $N_i/N_{i+1} = \mathrm{soc}(G/N_{i+1})$, hence, by Lemma 2, a normal subgroup of G which contains N_{i+1} either contains or is contained in N_i. This implies:

LEMMA 4: *If N is a nontrivial closed normal subgroup of G, then there exists $i \in \mathbb{N}$ such that $N_i \le N$. In particular, G is a just-infinite profinite group.*

Proof: Let N be a nontrivial closed normal subgroup of G; by the observation above, for any $i \in \mathbb{N}$, the normal subgroup NN_{i+1} either contains or is contained in N_i; in the first case $NN_{i+1} = NN_i$, in the second $N \le N_i$. Since N is closed, it must be $N = \bigcap_{i\in\mathbb{N}} NN_i$. There exists $i \in \mathbb{N}$ such that $N \not\le N_i$, otherwise $N \le \bigcap_{i\in\mathbb{N}} N_i = 1$. Let $j = \min\{i \in \mathbb{N} \mid N \not\le N_i\}$. For any $k \ge j$, we have $NN_{k+1} = NN_k$ hence $N = \bigcap_{i\in\mathbb{N}} NN_i = NN_j$, and this implies $N_j \le N$. ∎

As $d(G/N_i) = d(X_i) = 2$ for any $i \in \mathbb{N}$, we deduce that G *is a 2-generated profinite group.* However, we claim that G is not a PFG group. In order to prove this fact let us recall that Mann and Shalev proved that PFG groups can be characterized by the behaviour of the function $\mu_n(G)$ which is defined as the number of closed maximal subgroups of G with index n.

THEOREM 5 (Mann and Shalev [7] Theorem 4): *A finitely generated profinite group G is* **PFG** *if and only if G has polynomial maximal subgroup growth, i.e., there exists a constant c such that for all n, the number $\mu_n(G)$ is at most n^c.*

LEMMA 6: *The group G is not PFG.*

Proof: By construction, $G/N_{i+1} \cong X_{i+1} = (G_i)_{t_i}$ with $G_i = \mathrm{Alt}(m_i) \wr X_i$. Moreover, $\mathrm{soc}(X_{i+1}) = U_1 \times \cdots \times U_{t_i}$, where each U_j is a minimal normal subgroup of X_{i+1} and is isomorphic to $(\mathrm{Alt}(m_i))^{m_i}$. For any $j \in \{1, \ldots, t_i\}$ let $Y_j = \prod_{k \neq j} U_k$: this is a normal subgroup of X_{i+1} and $X_{i+1}/Y_j \cong G_i$. If we think of $\mathrm{Alt}(m_i)$ as a permutation group of degree m_i and we consider X_i as a regular permutation group of degree m_i, the wreath product $G_i = \mathrm{Alt}(m_i) \wr X_i$ with the product action is a primitive permutation group of degree $m_i^{m_i}$; this implies that for any $j \in \{1, \ldots, t_i\}$, the group X_{i+1} contains a maximal subgroup M_j such that $|X_{i+1} : M_j| = m_i^{m_i}$ and $\mathrm{core}_{X_{i+1}}(M_j) = Y_j$. Hence $\mu_{m_i^{m_i}}(G) \geq \mu_{m_i^{m_i}}(X_{i+1}) \geq t_i$. By Theorem 5, if G is PFG, then there exists a constant c such that for any $i \in \mathbb{N}$,

$$\left[\frac{\gamma(m_i!)^{m_i}}{m_i 2^{m_i+1}}\right] = t_i \leq \mu_{m_i^{m_i}}(G) \leq m_i^{cm_i}.$$

Since $\lim_{i \to \infty} m_i = \infty$ this is impossible. This proves that G is not PFG. ∎

We conclude this short paper with some remarks. The group

$$G = \prod_n (\mathrm{Alt}(n))^{n!/8}$$

admits crown-based powers with nonabelian socle as epimorphic images; furthermore, these crown-based powers may be taken of arbitrarily large size. Indeed if $k \geq 5$, then $k \leq k!/8$ and $(\mathrm{Alt}(k))_k = (\mathrm{Alt}(k))^k$ is an epimorphic image of G. In the same way let G be the just-infinite group constructed in this paper; for any positive integer k there exists i such that $k \leq t_i$ and G has a factor group isomorphic to the crown-based power $(G_i)_k$ of the primitive group G_i of size k. We conjecture that any finitely generated profinite group which is not PFG behaves in a similar way; more precisely our conjecture is: *if a finitely generated profinite group G is not PFG, then for any positive integer k there exists a primitive monolithic group L with nonabelian socle, such that the crown-based power L_k is an epimorphic image of G.*

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